Espace vectoriel

A(x)y' + b(x) y = 0 YESSM = K $\bullet\, y_1$

Equation à coefficient constants :

Ay'' + by' + cy = 0 (page 4 3.2.2)

y'' + 3y' – 4y = x $\bullet\, e^x$

intégration sans second membre:

1) ESSM $r^2 + 3r - 4 = 0 \quad \Rightarrow$ r = 1 r = 4

YESSM = $A \bullet e^x + B \bullet e^{-4x}$

2) $y_p = ?$ lecture de f(x)

$$f(x) = x \bullet e^x = [P(x) \bullet \cos(k \bullet x) + Q(x) \bullet \sin(k \bullet x)] \bullet e^{h \bullet x}$$

lecture : cos (k.x) = 1 $\Rightarrow$ k =0, h = 1, P(x) = x

test : h + $i \bullet k = 1$ racine simple $\Rightarrow$ solution du premier ordre

$$Y_p(x) = x \bullet e^x = x \bullet [R(x) \bullet \cos(0 \bullet x) + S(x) \bullet \sin(0 \bullet x)] \bullet e^{1 \bullet x} = x \bullet (a \bullet x + b) \bullet e^x = (a \bullet x^2 + bx) \bullet e^x$$

$$\begin{array}{c|r} -4 & y = (a \bullet x^2 + b \bullet x) \bullet e^x \\ 3 & y' = (a \bullet x^2 + b \bullet x + 2a \bullet x + b) \bullet e^x \\ 1 & y'' = (a \bullet x^2 + b \bullet x + 4a \bullet x + 2b + 2a) \bullet e^x \end{array}$$

y'' + 3y' – 4y = x $\bullet\, e^x$ = (10a.x + 5b +2a) $\bullet\, e^x$ 10a = 1, 5b + 2a = 0

$a = \dfrac{1}{10}$ $\qquad y_p(x) = (\dfrac{1}{10} \bullet x^2 - \dfrac{1}{25} \bullet x) \bullet e^x$

$b = -\dfrac{1}{25}$ $\qquad y_G(x) = A \bullet e^x + B \bullet e^{-4x} + (\dfrac{1}{10} \bullet x^2 - \dfrac{1}{25} \bullet x) \bullet e^x$

Ex : y'' + 3y' – 4y = x sin 4x

$$f(x) = x \bullet \sin 4x = [P(x) \bullet \cos(k \bullet x) + Q(x) \bullet \sin(k \bullet x)] \bullet e^{h \bullet x}$$

h = 0, k = 4 P(x) = 0 Q(x) = x polynome de degré de degré le plus haut

test h + ik = 4i non racine de l'équation caractéristique
donc

$$y_p(x) = x \bullet \sin 4x = [R(x) \bullet \cos(4 \bullet x) + S(x) \bullet \sin(4 \bullet x)] \bullet e^{0 \bullet x} = (a \bullet x + b) \cos(4 \bullet x) + (c \bullet x + c) \bullet \sin(4 \bullet x)$$

calculer a, b, c, d.

y'' + y' + y = x.sin x + 2 x^2 .cos x

1) ESSM $r^2 + r + 1 = 0$ $\Delta = -3$ r $= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \bullet x$

YESSEM $= e^{-\frac{x}{2}} \bullet [A \bullet \cos\frac{\sqrt{3}}{2} \bullet x + B \bullet \sin\frac{\sqrt{3}}{2} \bullet x]$

2) le second membre:

$f(x) = x \bullet \sin x + 2 \bullet x^2 \bullet \cos x = [P(x) \bullet \cos(k \bullet x) + Q(x) \bullet \sin(k \bullet x)] \bullet e^{h \bullet x}$

h =1 , k = 1 P(x) = $2 \bullet x^2$, Q(x) = x n = 2
rest h + k$\bullet i$ = i non racine
donc:
$y_p(x) = x \bullet [(a \bullet x^2 + b \bullet x + c) \bullet \cos x + (d \bullet x^2 + e \bullet x + f) \bullet \sin x]$
Si test racine de l'équation caractéristique racine double $\Rightarrow$ on multiplie par x (si racine double on mulitiplie par x^2)

Linéarité ⟶

y'' + y = x sin x + 2 cos 3x

1) ESSM (*) $r^2 + 1 = 0$ $r = \pm i$
YESSM =A cos x + B sin x

2) $y_{p1}(x) = x \bullet \sin x = [P(x) \bullet \cos(k \bullet x) + Q(x) \bullet \sin(k \bullet x)] \bullet e^{h \bullet x}$

h = 0 , k = 1, P(x) = 0, Q(x) = x

test h + i.k = i racine de l'équation caractéristique

donc : $y_{p1}(x) = x \bullet \sin x = x \bullet [(a \bullet x + b) \bullet \cos(x) + (c \bullet x + d) \bullet \sin(x)]$
calculer a, b, c, d

$y_{p2}(x) = 2 \bullet \cos(3 \bullet x) = [P(x) \bullet \cos(k \bullet x) + Q(x) \bullet \sin(k \bullet x)] \bullet e^{h \bullet x}$

h= 0, k = 3, P(x) = 2 Q(x) = 0, n = 0

$y_{p2}(x) = x \bullet \sin x = [(a \bullet x) \bullet \cos(3 \bullet x) + (b \bullet x) \bullet \sin(3 \bullet x)]$ calcul a, b
$Y_p = Y_{p1} + Y_{p2}$ et $Y_G = Y_{ESSM} + Y_P$

Utilisation de la méthode des variations des constantes :

Ex : $y'' - y = \frac{1}{x}$ ESSM (*) $r^2 - 1 = 0 \quad r = \pm 1$

$$Y_{ESSM} = A \bullet e^{x} + B \bullet e^{-x}$$

On cherche une solution de (E) sous la forme $y_{P(x)} = A(x) \bullet e^{x} + B(x) \bullet e^{-x}$

Dérivation d'un produit :

$$-1 \quad = y_{p(x)} = \qquad A(x) \bullet e^{x} + B(x) \bullet e^{-x}$$

$$0 \quad = y'_{p(x)} = \quad A(x) \bullet e^{x} - B(x) \bullet e^{-x} + A'(x) \bullet e^{x} - B'(x) \bullet e^{-x}$$

imposé à zéro (revient à diminuer le

$$1 \quad = y''_{p(x)} = \quad A(x) \bullet e^{x} + B(x) \bullet e^{-x} + A'(x) \bullet e^{x} - B'(x) \bullet e^{-x}$$

nombre de solution

$$\begin{cases} A'(x) \bullet e^{x} - B'(x) \bullet e^{-x} = \frac{1}{x} \\ A'(x) \bullet e^{x} + B'(x) \bullet e^{-x} = 0 \end{cases}$$

$$2 \bullet A'(x) \bullet e^{x} = \frac{1}{x} \quad \Rightarrow A'(x) = \frac{e^{-x}}{2 \bullet x} \quad \Rightarrow A(x) = \int \frac{e^{-x}}{2 \bullet x} \bullet dx + cte$$

$$2 \bullet B'(x) \bullet e^{-x} = -\frac{1}{x} \quad \Rightarrow B'(x) = -\frac{e^{-x}}{2 \bullet x} \quad \Rightarrow B(x) = -\int \frac{e^{-x}}{2 \bullet x} \bullet dx + cte$$

On cherche une solution particulière.

Fonction réelles à plusieurs variables.

$f(x,y) = x^3 \bullet y^2 \quad \frac{df}{dx}(x,y) = 3 \bullet x^2 \bullet y^2$ avec y constant

$f(x,y) = x^3 \bullet y^2 \quad \frac{df}{dy}(x,y) = 3 \bullet x^2 \bullet y^2$ avec x constant

$$\frac{\partial^2 f}{dx \bullet dy}(x,y) = \frac{\partial}{\partial x} \bullet [\frac{\partial f}{\partial x}](x,y) = \frac{\partial}{\partial x}[2 \bullet x^2 \bullet y] = 6 \bullet x^2 \bullet y$$

$$\frac{\partial^2 f}{dy \bullet dx}(x,y) = \frac{\partial}{\partial y} \bullet [\frac{\partial f}{\partial y}](x,y) = \frac{\partial}{\partial x}[3 \bullet x^2 \bullet y] = 2 \bullet 3 \bullet x^2 \bullet y$$

Démontrer :

$\overrightarrow{rot} \bullet (\overrightarrow{grad} \bullet U) = \vec{0}$

$div \bullet (\overrightarrow{vot} \bullet \vec{V}) = 0$

$div \bullet (\overrightarrow{grad} \bullet U) = \Delta U$

$$\overrightarrow{grad} \bullet U = \left| \begin{array}{lcl} \dfrac{du}{dx} & = & P \\ \dfrac{du}{dy} & = & Q \\ \dfrac{du}{dz} & = & R \end{array} \right. \qquad \overrightarrow{rot} \bullet (\overrightarrow{grad} \bullet U) = \left| \begin{array}{lcl} \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} & = \dfrac{d^2U}{dy \bullet dz} - \dfrac{d^2U}{dz \bullet dy} & = 0 \\ \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x} & = \dfrac{d^2U}{dz \bullet dx} - \dfrac{d^2U}{dz \bullet dx} & = 0 \\ \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} & = \dfrac{d^2U}{dx \bullet dy} - \dfrac{d^2U}{dy \bullet dx} & = 0 \end{array} \right.$$

$$\overrightarrow{rot} \bullet \vec{V} = \left| \begin{array}{l} \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x} \\ \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \end{array} \right. \Rightarrow div \bullet (\overrightarrow{rot} \bullet \vec{V}) = \overrightarrow{rot} \bullet \vec{V} = \left| \begin{array}{l} \dfrac{\partial}{\partial x} \bullet \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial}{\partial y} \bullet \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x} \\ \dfrac{\partial}{\partial z} \bullet \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \end{array} \right| = 0$$

$$\Delta U(M) = \frac{\partial^2 U}{\partial^2 x}(M) + \frac{\partial^2 U}{\partial^2 y}(M) + \frac{\partial^2 U}{\partial^2 z}(M)$$

$$\vdots$$

$$div \bullet (\overrightarrow{grad} \bullet U = \Delta U$$

nabla $\longrightarrow$ $\vec{\nabla} = \left. \begin{array}{l} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{array} \right. \qquad \vec{V} = \left| \begin{array}{l} P \\ Q \\ R \end{array} \right.$

$$\overrightarrow{grad} \bullet U = \vec{\nabla} \bullet U$$

$$div \bullet \vec{V} = \vec{\nabla} \bullet \vec{V}$$

$$\overrightarrow{rot} \bullet \vec{V} = \vec{\nabla} \bullet \vec{V}$$

$$\overrightarrow{rot} \bullet \vec{V} = \vec{\nabla} \wedge \vec{V}$$

$$\Delta U = (\vec{\nabla} \bullet \vec{\nabla}) \bullet U$$

1ère propriété : colinéaire à $\vec{\nabla}$

$$\overrightarrow{rot} \bullet (\overrightarrow{grad} \bullet U) = \vec{\nabla} \wedge (\vec{\nabla} \bullet U) = \vec{0}$$

2ème propriété : orthogonale à $\vec{\nabla}$

$$div \bullet (\overrightarrow{rot} \bullet \vec{V}) = \vec{\nabla} \bullet (\vec{\nabla} \wedge \vec{V})$$

3ème propriété :

$$div \bullet (\overrightarrow{grad} \bullet U) = \vec{\nabla} \bullet (\vec{\nabla} \bullet U) = (\vec{\nabla} \bullet \vec{\nabla}) \bullet U = \Delta \bullet U$$

Remarques :

F $\begin{cases} R \to R \\ x \to f(x) \end{cases}$ x $\begin{cases} R \to R \\ u \to x = x(u) \end{cases}$ F $\begin{cases} R \to R \\ u \to f[x(u)] \end{cases}$ F'(u) = f'[x(u)] $\bullet\, x'(u)$

$$\frac{dF}{du} = \frac{df}{\cancel{dx}} \bullet \frac{\cancel{dx}}{du}$$

f $\begin{cases} R^2 \to R^2 \\ (x,y) \to f(x,y) \end{cases}$

x $\begin{cases} R^2 \to R^2 \\ (u,v) \to x = x(u,v) \end{cases}$

y $\begin{cases} R^2 \to R^2 \\ (u,v) \to y = y(u,v) \end{cases}$

F $\begin{cases} R^2 \to R \\ (u,v) \to f[x(u,v), y(u,v)] \end{cases}$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \bullet \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \bullet \frac{\partial y}{\partial u}$$ (pas le droit de simplifier)

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \bullet \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \bullet \frac{\partial y}{\partial v}$$

matrice jacobienne:

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

transformé de la matrice jacobienne :

$$t_J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{matrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{matrix}$$

$$\begin{pmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial F}{\partial v} \end{pmatrix} = t_J = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

2.2

$$df_{M0}\begin{cases} R^2 \to R \\ (h,k) \to \frac{\partial f}{\partial x}(M_0)h + \frac{\partial f}{\partial y}(M_0) \bullet k \end{cases}$$

Ex : ''x'' $\begin{cases} R^2 \to R \\ (x,y) \to x \end{cases}$ ''y'' $\begin{cases} R^2 \to R \\ (x,y) \to y \end{cases}$

$$dx_{M0}\begin{cases} R^2 \to R \\ (x,y) \to 1 \bullet h + 0 \bullet k = h = dx_{M0} \end{cases} \qquad dx_{M0}\begin{cases} R^2 \to R \\ (x,y) \to 0 \bullet h + 1 \bullet k = k = dy_{M0} \end{cases}$$

$\forall (h,k) \in R$

$$df_{M0}(\text{h,k}) = \frac{\partial f}{\partial x}(M_0)h + \frac{\partial f}{\partial y}(M_0) \bullet k = \frac{\partial f}{\partial x}(M_0) \bullet dx_{M0}(h,k) + \frac{\partial f}{\partial y}(M_0)(h,k)$$

constante

appli

donc

$$df_{M0} = \frac{\partial f}{\partial x}(M_0) \bullet dx_{M0} + \frac{\partial f}{\partial y}(M_0) \bullet dy_{M0}$$

égalité d'application Linéaires.

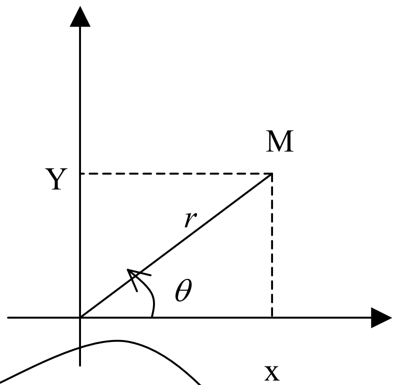
$$df = \frac{\partial f}{\partial x} \bullet dx + \frac{\partial f}{\partial y} \bullet dy$$

infinité d'égalités d'application linéaires

2.2.2 (page5

coordonnées cartésiennes → coordonnée polaire

bijective :



x = $r \bullet \cos\theta$
y = $r \bullet \sin\theta$

$r \geq 0$
$\theta \in [0,2\pi]$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial r} \bullet \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \bullet \frac{\partial \theta}{\partial u}$$

dérivié partielle

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial r} \bullet \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \bullet \frac{\partial \theta}{\partial y}$$

exprimer r en fonction de x et y

r $=\sqrt{x^2+y^2} \quad \theta = \arctan\frac{y}{x}$

$\frac{dr}{dx} = \frac{2x}{2\sqrt{x^2+y^2}}$ pas bon car exprimé en fonction de x et de y

d'où ;

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r \bullet \cos\theta$

$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = r \bullet \cos\theta$

$$dx = \frac{\partial x}{\partial r} \bullet dr + \frac{\partial y}{\partial \theta} \bullet d\theta = \cos\theta \bullet dr - \sin\theta \bullet d\theta$$

$$dy = \frac{\partial y}{\partial r} \bullet dr + \frac{\partial y}{\partial \theta} \bullet d\theta = \sin\theta \bullet dr + \cos\theta \bullet d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -r\bullet\sin\theta \\ \sin\theta & r\bullet\cos\theta \end{pmatrix}}_{J} \begin{pmatrix} dr \\ d\theta \end{pmatrix} \qquad \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \underbrace{\begin{bmatrix} \; \end{bmatrix}}_{J^{-1}} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \Delta_J = r\bullet\cos^2\theta + r\bullet\sin^2\theta = r$$

$$\tilde{J} = \begin{pmatrix} r\bullet\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad J^{-1} = \frac{1}{r}\bullet\begin{pmatrix} r\bullet\cos\theta & r\bullet\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

$$dr = \cos\theta\bullet dx + \sin\theta\bullet dy = \frac{\partial r}{\partial x}\bullet dx + \frac{\partial r}{\partial y}\bullet dy$$

$$d\theta = -\frac{\sin\theta}{r}\bullet dx + \frac{\cos\theta}{r}\bullet dy = \frac{\partial\theta}{\partial x}\bullet dx + \frac{\partial\theta}{\partial y}\bullet dy$$

$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta$

$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

ex2 : $x \bullet \frac{\partial f}{\partial x} + y \bullet \frac{\partial f}{\partial y} = 0$ $\quad$ $\begin{array}{l} x = r \bullet \cos\theta \\ y = r \bullet \sin\theta \end{array}$ on va confondre F et f

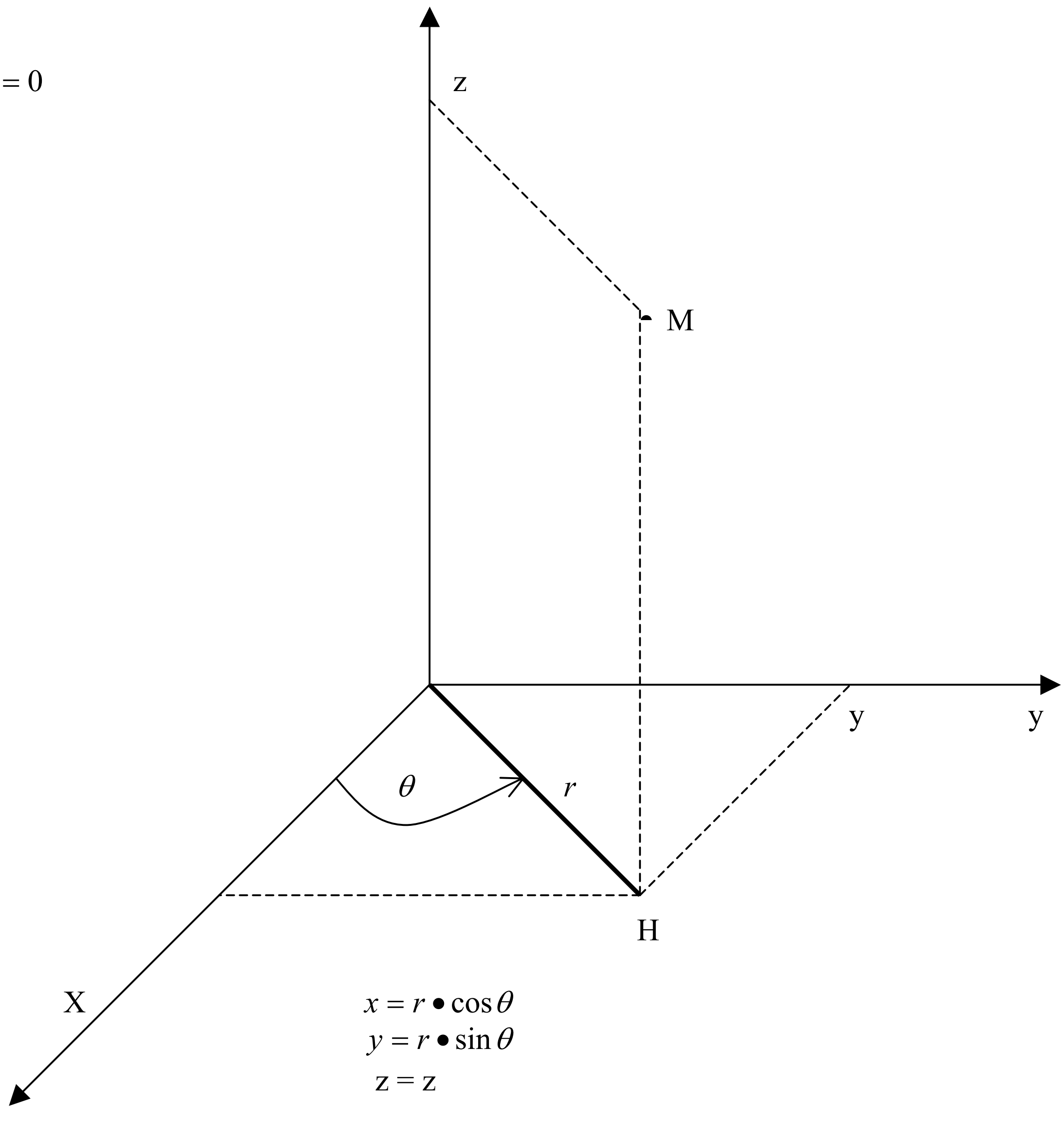
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \bullet \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \bullet \frac{\partial \theta}{\partial x} = \frac{\partial f}{\partial r} \bullet \cos\theta + \frac{\partial f}{\partial \theta} \bullet \frac{\sin\theta}{r}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \bullet \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \bullet \frac{\partial \theta}{\partial y} = \frac{\partial f}{\partial r} \bullet \sin\theta + \frac{\partial f}{\partial \theta} \bullet \frac{\cos\theta}{r}$$

$$r \bullet \cos\theta \bullet [\frac{\partial f}{\partial r} \bullet \cos\theta - \frac{\partial f}{\partial \theta} \bullet \frac{\sin\theta}{r}] + r \bullet \sin\theta \bullet [\frac{\partial f}{\partial r} \bullet \sin\theta + \frac{\partial f}{\partial \theta} \bullet \frac{\cos\theta}{r}]$$

$$r \bullet \frac{\partial f}{\partial r} \bullet (\cos^2\theta + \sin^2\theta) = 0$$

$$r \bullet \frac{\partial f}{\partial r} = 0$$



$$x = r \bullet \cos\theta$$
$$y = r \bullet \sin\theta$$
$$z = z$$

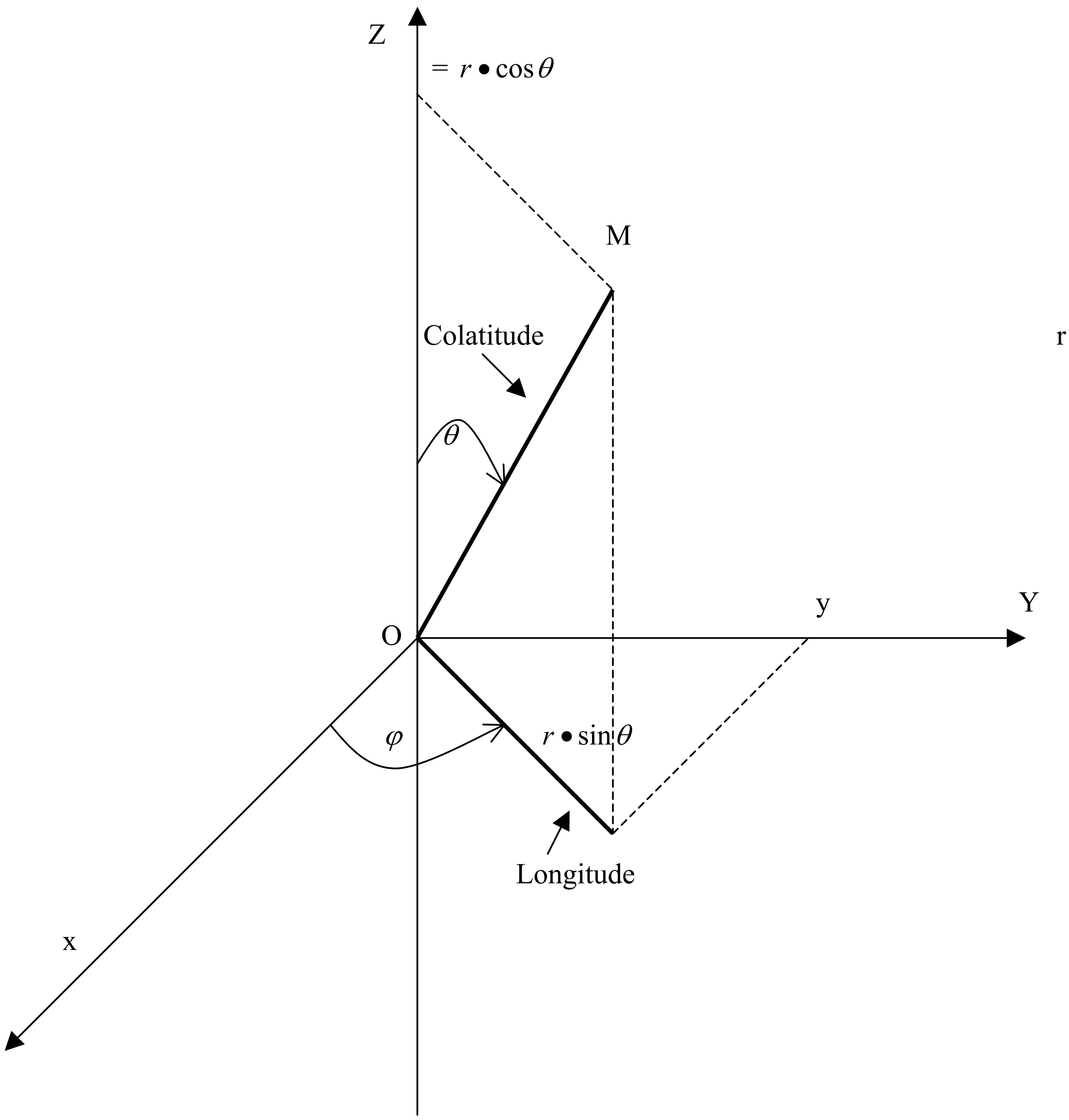
$$J = \begin{pmatrix} \cos\theta & -r \bullet \sin\theta & 0 \\ \sin\theta & r \bullet \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{array} \qquad \tilde{J} = \begin{pmatrix} r \bullet \cos\theta & -\sin\theta & 0 \\ r \bullet \sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix}$$

det J = r

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ r & \theta & z \end{matrix}$$

$$J^{-1} = \frac{1}{r} \bullet \begin{pmatrix} r \bullet \cos\theta & r \bullet \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} \begin{matrix} r \\ \theta \\ z \end{matrix}$$

$$\begin{matrix} x & y & z \end{matrix}$$



$x = r \bullet \sin\theta \bullet \cos\varphi$

$y = r \bullet \sin\theta \bullet \sin\varphi$

$z = r \bullet \cos\theta \qquad \theta \in [0, \pi] \qquad \varphi = [0, 2\pi] \qquad r \geq 0$